Relations and Functions

Q1. Among the relations

$\mathbf{S} = \left\{(\mathbf{a}, \mathbf{b}): \mathbf{a}, \mathbf{b} \in \mathbb{R} - \{\mathbf{0}\}, \mathbf{2} + \frac{\mathbf{a}}{\mathbf{b}} > \mathbf{0}\right\}$ **and**

$\mathbf{T} = \{(\mathbf{a}, \mathbf{b}): \mathbf{a}, \mathbf{b} \in \mathbb{R}, \mathbf{a}^2 - \mathbf{b}^2 \in \mathbb{Z}\}$,

(1) neither S nor T is transitive
(2) S is transitive but T is not
(3) T is symmetric but S is not
(4) both S and T are symmetric

Ans. (3)

Basic Definitions of Relations

A relation R on a set A is said to have the following properties:

- Symmetric: If (a, b) ∈ R ⇒ (b, a) ∈ R
- Transitive: If (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R
- Reflexive: If (a, a) ∈ R for all a ∈ A (not asked in this question)

Step-by-Step Solution

Analyzing Relation S

S = {(a, b) ∈ ℝ \ {0} × ℝ \ {0} : 2 + a/b > 0}
Let's test if S is symmetric:

Choose a = 1, b = -1:
2 + 1/(-1) = 1 > 0 ⇒ (1, -1) ∈ S
Now check (-1, 1): 2 + (-1)/1 = 1 > 0 ⇒ (-1, 1) ∈ S
⇒ S might seem symmetric

Try a = -5, b = 2:
2 + (-5)/2 = -0.5 < 0 ⇒ (−5, 2) ∉ S
Now test (2, -5): 2 + 2/(-5) = 2 - 0.4 = 1.6 > 0 ⇒ (2, -5) ∈ S
So (2, -5) ∈ S but (−5, 2) ∉ S ⇒ S is not symmetric ✗

Now test if S is transitive:
Assume (a, b) ∈ S and (b, c) ∈ S. Does it imply (a, c) ∈ S?

Try a = 1, b = -1, c = -2:
(1, -1): 2 + 1/(-1) = 1 > 0 ⇒ ∈ S
(-1, -2): 2 + (-1)/(-2) = 2.5 > 0 ⇒ ∈ S
(1, -2): 2 + 1/(-2) = 1.5 > 0 ⇒ ∈ S ⇒ In this case transitive

Now try a = -5, b = 2, c = -1:
(-5, 2): 2 + (-5)/2 = -0.5 ⇒ ∉ S ⇒ Transitivity fails as condition is not preserved in general
⇒ S is not transitive ✗

Analyzing Relation T

T = {(a, b) ∈ ℝ × ℝ : a² − b² ∈ ℤ}
Check symmetry:
If (a, b) ∈ T ⇒ a² − b² ∈ ℤ
Then (b, a) ⇒ b² − a² = −(a² − b²) ∈ ℤ since integers are closed under negation
⇒ T is symmetric ☑

Check transitivity:
Assume (a, b) ∈ T and (b, c) ∈ T
Then a² − b² ∈ ℤ and b² − c² ∈ ℤ
Add: a² − b² + b² − c² = a² − c² ∈ ℤ ⇒ (a, c) ∈ T
⇒ T is transitive ☑

Conclusion

- S is neither symmetric nor transitive ✗
- T is symmetric ☑ (and also transitive, though not asked)

Therefore, the correct answer is:

✔ (3) T is symmetric but S is not

Q2. If the domain of the function $f(x) = \frac{[x]}{1+x^2}$, where $[x]$ is greatest integer $\leq x$, is $[2, 6)$, then its range is

(1) $\left(\frac{5}{26}, \frac{2}{5}\right]$
(2) $\left(\frac{5}{37}, \frac{2}{5}\right] - \left\{\frac{9}{29}, \frac{27}{109}, \frac{18}{89}, \frac{9}{53}\right\}$
(3) $\left(\frac{5}{37}, \frac{2}{5}\right]$
(4) $\left(\frac{5}{26}, \frac{2}{5}\right] - \left\{\frac{9}{29}, \frac{27}{109}, \frac{18}{89}, \frac{9}{53}\right\}$